Observables of Euclidean Supergravity

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Abstract

The set of constraints under which the eigenvalues of the Dirac operator can play the role of the dynamical variables for Euclidean supergravity is derived. These constraints arise when the gauge invariance of the eigenvalues of the Dirac operator is imposed. They impose conditions which restrict the eigenspinors of the Dirac operator.

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During the last years, the Dirac operator has become a very powerful tool for studying the geometrical properties of the manifolds as well as the fundamental physics that takes place on them. Not long ago, Connes showed, in the context of noncommutative geometry, that the Dirac operator contains full information about the geometry of space-time [1]. It turned out that this property makes the Dirac operator suitable for describing the dynamics of the general relativity and that, at least in principle, the Dirac operator can be used instead of the metric to describe the gravitational field [2]-[9]. However, there are still some major problems that must be solved before this point of view be totally accepted. One of the most important ones is raised by the fact that the spectrum of the Laplace type operators (like the squared Dirac operator) cannot uniquely determine the topology and the geometry of a four-dimensional Riemannian manifold (a detailed analysis on this topic can be found in [3]). Another problem arises when Riemannian manifolds without boundary are considered. They provide only an idealization of the manifolds encountered in physics since boundary terms play a crucial role in many important physical phenomena, as for example in determining the black-hole entropy from a perturbative evaluation of the path-integral for the partition function [4]. In the spectral geometry approach to Euclidean gravity, a major role is played by the eigenvalues λ^n's of the Dirac operator D which are diffeomorphism-invariant functions of the geometry and thus can be considered as the observables of general relativity. In a very recent paper, Landi and Rovelli expressed the Poisson bracket of λ^n's in terms of the components of the energy-momentum tensor of the corresponding eigenspinor, and derived the Einstein equations from a spectral action with no cosmological term [5]. This could be a new way to think of quantum gravity, but this method works at the moment being only for the Euclidean theory. There are, however, several attempts to implement

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the method in the Lorentzian case [10, 11]. Nevertheless, the Euclidean case by itself is quite interesting and it is the only one for which the ellipticity of the some differential operators can be ensured. Also, many interesting problems can be formulated in a well defined form only in the Euclidean setting. In gravity, the Euclidean case allows to derive several interesting solutions of the Einstein equations, as Euclidean wormholes which, together with the minisuperspace technique, were invoked to explain constants in nature, as the vanishing cosmological constant [18]. Another interesting problem is that in some circumstances spacetime might change signature and become Euclidean. That shows that it would be worthwhile to investigate what happens when physical objects pass from one region to the other one. These examples motivate enough the study of the Euclidean systems and in this context it is natural to ask whether the description of the space-time by means of the eigenvalues of the Dirac operator can be extended further to include the supersymmetric case. If a supersymmetric partner of the metric is considered, and the local supersymmetry is imposed, we are led to Euclidean supergravity. This kind of systems has been extensively studied lately mainly into the frame of path integral quantization of supergravity with a stress on the problem of the boundary conditions which are to be imposed on the fermions [12, 14] (see also [17]). In this paper it is addressed the question whether the eigenvalues of the Dirac operator can be used as observables for Euclidean supergravity.

Consider Euclidean minimal supergravity on a compact D=4 (spin) manifold with no boundary. The graviton is represented in the tetrad formalism by the fields $e_\mu^a(x)$; $\mu = 1, \cdots, 4$ are space-time indices and $a = 1, \cdots, 4$ are internal Euclidean indices, raised and lowered by the Euclidean metric δ_{ab}. The metric field is $g_{\mu\nu}(x) = e_\mu^a(x)e_{\nu a}(x)$. The gravitino is represented by a Euclidean spin-vector field $\psi_\mu(x)$. In order to have " Euclidean Majorana spinors" and to

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maintain the correct number of degrees of freedom required by a supersymmetric theory, the adjoint spinors are defined via Majorana conjugation relation $\bar{\psi} = \psi^T C$. This encounters the problem posed by the fact that there is no Majorana spinor representation of $SO(4)$ and ensures the theory is supersymmetric [15](for other discussions on the "Euclidean Majorana spinors" see also [13, 16, 17]). The spin connection $\omega_{\mu ab}(e, \psi)$ is defined as

$$\omega_{\mu ab}(e, \psi) = \overset{\circ}{\omega}_{\mu ab}(e) + K_{\mu ab}(\psi) \tag{1}$$

where $\overset{\circ}{\omega}_{\mu ab}(e)$ is the usual spin-connection of gravity. In units such that $8\pi G = 1$ the following relations are assumed true

$$\overset{\circ}{\omega}_{\mu ab}(e) = \frac{1}{2}e^\mu_a(\partial_\mu e_{b\nu} - \partial_\nu e_{b\mu}) + \frac{1}{2}e^\rho_a e^\sigma_b \partial_\sigma e_{\rho c} e^c_\mu - (a \leftrightarrow b) \tag{2}$$

$$K_{\mu ab}(\psi) = \frac{i}{4}(\bar{\psi}_\mu \gamma_a \psi_b - \bar{\psi}_\mu \gamma_b \psi_a + \bar{\psi}_b \gamma_\mu \psi_a). \tag{3}$$

As usual in the supersymmetric case, there are two covariant derivative acting on space-time tensors and space-time spinors, respectively. The minimal covariant derivative, when acts on vectors, for example, is expressed in terms of Christoffel symbols as

$$D_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\sigma} V^\sigma \tag{4}$$

and the non-minimal covariant derivative which acts on spinors reads as follows

$$\mathcal{D}_\mu \phi = \partial_\mu \phi - \frac{i}{2}\omega_{\mu ab}(e, \psi)\sigma^{ab}\phi \tag{5}$$

where $\sigma^{ab} = \frac{1}{4}[\gamma^a, \gamma^b] = i\Sigma^{ab}$ and γ^a's form an Euclidean representation of the Clifford algebra C_4: $\{\gamma^a, \gamma^b\} = \delta^{ab}$.

The phase space of the theory is defined as the space of the solutions of the equations of motion, modulo the gauge transformations [5, 20]. The gauge transformations are 4D diffeomor-

phisms, local $SO(4)$ rotations and the local $N = 1$ supersymmetry. Then the phase space, which is covariantly defined, is the space of all e, ψ that are solutions of the equations of motion modulo diff's, internal rotations and local supersymmetry. Because the phase space is defined over the solutions of the equations of motion it is sufficient to consider only on-shell supersymmetry. In this case, the supersymmetric algebra closes over only graviton and gravitino. Off-shell, the supersymmetric algebra usually requires six more bosonic fields since there is a mismatch of the fermionic and bosonic degrees of freedom. As in [5] the observables of the theory are the functions on the phase space. For Euclidean gravity the eigenvalues λ^n's of D define a discrete real family of real-valued functions on the space of all tetrads and for every n the function $\lambda^n(e)$ is invariant under diff's and under internal rotations. Therefore they are well defined on the phase space and they are observables of general relativity. For the present case one has to find what are the conditions under which λ^n's are gauge invariant such that they can be used as observables of supergravity. The presence of the gravitino and the requirement of the local susy give a non-trivial solution to this problem.

In the present case the Dirac operator is defined as

$$D = i\gamma^a e_a^\mu (\partial_\mu + \omega_{\mu bc}(e, \psi)\gamma^b\gamma^c) \tag{6}$$

and it acts on the Euclidean spinors defined on the manifold. It is possible to construct the Dirac operator such that it is self-adjoint on the Hilbert space of spinor fields with the scalar product defined as in [5]

$$< \psi, \phi >= \int d^4x \sqrt{g}\psi^*(x)\phi(x) \tag{7}$$

where ψ^* represents the complex conjugate. Now D is different from the usual curved Dirac

operator which is denoted by $\overset{\circ}{D}$ and is defined in [5], because of the term $K_{\mu ab}$ that enters $\omega_{\mu ab}$ and which is required by the supersymmetry. However, D may also have a discrete spectrum of eigenvalues and eigenvectors. This is not a trivial problem and one can think of its possible resolution by considering that the operator $\Omega = i\gamma^a e^{\mu}_a \omega_{\mu bc}(e, \psi)\gamma^b \gamma^c$ is a perturbation of D from $\overset{\circ}{D}$ and that it controls the spectrum of the former. A detailed analysis of this operator is not the purpose of this paper and the reader is referred to [19] for a deeper discussion of this matter. In what follows it is assumed that D admits a discrete spectrum of eigenvalues and eigenspinors and thus we can write

$$D\chi^n = \lambda^n \chi^n \tag{8}$$

Then λ^n's define a discrete family on the space of all gravitons and gravitinos denoted by $\mathcal{F}$. As in the case of gravity, λ^n's may presumably coordinate the space of orbits of the gauge group in the space $\mathcal{F}$ [19]. However, even if the above conditions hold, λ^n's might fail to be invariant under the gauge group. In fact, the gauge invariance of λ^n's impose additional constraints on $\mathcal{F}$ as well as on the eigenspinors of D.

To see this, consider the variation of any λ^n under diff's. This variation is generated by an arbitrary small vector field $\xi = \xi^{\mu}\partial_{\mu}$. The vector field generates infinitesimal transformations by the mean of the Lie derivative. Since $D = D(e, \psi)$ the eigenvalues λ^n depend on the independent variables e^a_{μ} and ψ^{α}_{μ}. Then we can write for the Lie derivative

$$\delta \lambda^n = \frac{\delta \lambda^n}{\delta e^a_{\mu}}\xi^{\nu}\partial_{\nu}e^a_{\mu} + \frac{\delta \lambda^n}{\delta \psi^{\alpha}_{\mu}}\xi^{\nu}\partial_{\nu}\psi^{\alpha}_{\mu} \tag{9}$$

$$= <\chi^n|\frac{\delta}{\delta e^a_{\mu}}D|\chi^n> \xi^{\nu}\partial_{\nu}e^a_{\mu} + <\chi^n|\frac{\delta}{\delta \psi^{\alpha}_{\mu}}D|\chi^n> \xi^{\nu}\partial_{\nu}\psi^{\alpha}_{\mu} \tag{10}$$

A simple algebra shows that, for the variation (10) to vanish, the following set of equations

should hold

$$\mathcal{T}_a^{n\mu}\partial_\nu e_\mu^a - \Gamma_a^{n\mu}\partial_\nu\psi_\mu^\alpha = 0 \tag{11}$$

where $\mathcal{T}_a^{n\mu} = T_a^{n\mu} + K_a^{n\mu}$ where $T_a^{n\mu}$ is the "energy-momentum tensor" of the spinor χ^n [5, 21] and $K_a^{n\mu} = <\chi^n|i\gamma_a K_{bc}^\mu(\psi)\sigma^{bc}|\chi^n>$. The last term comes from the derivation of spin-connection with respect to the spinor field and is given by

$$\Gamma_a^{n\mu} = \frac{i}{4}\int\sqrt{e}\chi^{n*}\gamma^a e_a^\nu[\bar{\psi}_\nu^\beta(\gamma_b)_{\alpha\beta}e_c^\mu - \bar{\psi}_\nu^\beta(\gamma_c)_{\alpha\beta}e_b^\mu + \bar{\psi}_b^\beta(\gamma_\nu)_{\alpha\beta}e_c^\nu]\sigma^{bc}\chi^n. \tag{12}$$

In a similar manner the invariance of the eigenvalues under the $SO(4)$ raise new constraints. In this case e_μ^a transforms under rotations as a vector in the upper index while ψ_μ^α belongs to the spinorial representation of $SO(4)$ ($\delta e_\mu^a = \theta^{ab}e_{b\mu}, \delta\psi_\mu^\alpha = \theta^{ab}(\sigma_{ab})_\beta^\alpha\psi_\mu^\beta$). Performing the same steps as in the case of the invariance under diff's one gets the following constraints

$$\mathcal{T}_a^{n\mu}e_{b\mu} + \Gamma^{n\mu}\sigma_{ab}\psi_\mu = 0 \tag{13}$$

In order to investigate the local susy invariance we consider the following on-shell local susy transformations

$$\delta e_\mu^a = \frac{1}{2}\bar{\epsilon}\gamma^a\psi_\mu \qquad \delta\psi_\mu = \mathcal{D}_\mu\epsilon \tag{14}$$

where $\epsilon(x)$ is an infinitesimal spinor field for which the $\bar{\epsilon} = \epsilon^T C$ is true. Under (14) the spin connection transforms as:

$$\delta\omega_\mu^{ab} = A_\mu^{ab} - \frac{1}{2}e_\mu^b A_c^{ac} + \frac{1}{2}e_\mu^a A_c^{bc} \tag{15}$$

where

$$A_a^{\mu\nu} = \bar{\epsilon}\gamma_5\gamma_a\mathcal{D}_\lambda\psi_\rho\epsilon^{\nu\mu\lambda\rho} \tag{16}$$

The vanishing of the variation of the eigenvalues of the Dirac operator lead to further constraints

$$\mathcal{T}_a^{n\mu}\bar{\epsilon}\gamma^a\psi_\mu + \Gamma^{n\mu}\mathcal{D}_\mu\epsilon = 0. \tag{17}$$

The set of equations (11),(13),(17) define the necessary conditions for λ^n's be invariant under the gauge group. These conditions represent a new type of constraints on the space $\mathcal{F}$ of all possible supermultiplets. Furthermore, as one can see by simply inspecting the relations (11), (13), (17) the equations are not independent. Therefore the geometry of the constrained surface is quite complicated and this makes the quantization problem highly non-trivial. To deal with this problem one has to work with BV-BRST or BFV-BRST quantization method developed to handle such situations [19].

The above constraints are not the only ones arising in this theory. If the equation (8) is subjected to the infinitesimal transformations of the gauge group and the variation of $\delta\lambda^n$ vanishes as required previously we get

$$\delta D\chi^n = (\lambda^n - D)\delta\chi^n \tag{18}$$

In the case when the above variations are induced by diff's relation (18) reads

$$\{[b^\mu(\xi) - c(\lambda\xi)^\mu]\partial_\mu + f(\xi)\}\chi^n = 0 \tag{19}$$

where the following notations are used

$$b^\mu(\xi) = i\gamma^a b_a^\mu(\xi) \ , \quad b_a^\mu(\xi) = \xi^\nu\partial_\nu e_a^\mu - e_a^\nu\partial_\nu\xi^\mu - 2e_a^\nu\xi^\mu\omega_{\nu bc}\sigma^{bc} \tag{20}$$

$$c(\lambda,\xi)^\mu = (\lambda^n - D)\xi^\mu \ , \quad f(\xi) = i\gamma^a\xi^\nu\partial_\nu(e_a^\mu\omega_{\mu bc})\sigma^{bc}. \tag{21}$$

A similar relation occurs when in (18) the rotations are considered. In this case the spin connec-

tion transforms as the gauge field for rotations

$$\delta\omega_{\mu ab} = i[\theta\sigma, \omega_{\mu ab}] - i\partial_\mu \theta\sigma M_{ab} \tag{22}$$

where $\theta_{ab} = -\theta_{ba}$ parameterize an infinitesimal $SO(4)$ rotation and $\theta\sigma = \theta_{ab}\sigma_{ab}$. Since χ^n transforms in the unitary spinor representation of $SO(4)$ we can write

$$\delta\chi^n = i\theta\sigma\chi^n \tag{23}$$

Using these transformations, the equation (18) becomes

$$[\theta_a^a D - g(\theta) + h(\theta)]\chi^n = 0 \tag{24}$$

where the following notations are used

$$g(\theta) = [\gamma^c e_c^\mu ([\theta\sigma, \omega_{\mu ab}] - \partial_\mu \theta\sigma M_{ab})]\sigma^{ab} \tag{25}$$

$$h(\theta) = i(\lambda^n - D)\theta\sigma \tag{26}$$

Now if the case of N=1 local supersymmetry is considered, it must be noticed that χ^n's are unaffected by this symmetry and thus the left-hand side of equation (18) vanishes. That leads eventually to the equation

$$[j_a^\mu(\epsilon)\partial_\mu + k_a(\epsilon) + l_a]\chi^n = 0 \tag{27}$$

where the notations used above are

$$j_a^\mu(\epsilon) = \frac{1}{2}\gamma_a \bar{\epsilon}\psi^\mu \ , \quad k_a(\epsilon) = \frac{1}{2}\gamma_a \bar{\epsilon}\psi^\mu \omega_{\mu cd}\sigma^{cd} \tag{28}$$

$$l_a = e_a^\mu[B_{\mu cd} - \frac{1}{2}e_{\mu d}B_{ec}^e + \frac{1}{2}e_{\mu c}B_{ed}^e]\sigma^{cd}. \tag{29}$$

The final relations (19), (24), (27) can be interpreted as constraints on the eigenspinors of the Dirac operator. They depend on the supermultiplets as well as on the eigenvalues of the Dirac

operator and are direct consequences of the invariance of λ^n's under the gauge group of the problem. These supplementary constraints complicate the description of the covariant phase space as well as a possible tentative to formulate the quantum problem in this case [19].

In summary, I have discussed the possibility of considering the eigenvalues of the Dirac operator as observables of Euclidean supergravity. The invariance of the eigenvalues under the gauge group of the problem imposes severe constraints on the space of the supermultiplets. The form of these constraints was completely determined. The constraints (11),(13),(17) involve both the derivatives of the gravitons and the gravitinos as well as the integral of gravitinos. Since these equations form a system non-independent integral-differential equations, at this moment one can only speculate on their solutions and their form is unknown to the author yet. However, the relations (11),(13),(17) define a complicate surface on the phase space and thus one has to use BV or BFV methods of quantization in order to construct the quantum problem of this system. The eigenspinors get themselves constrained too, from the requirement that the equations that define the eigenvectors and eigenvalues satisfy the gauge group symmetry. The corresponding equations are (19), (24), (27) and they hold whenever the constraints (11),(13),(17) are satisfied, i.e. the gravitons and gravitinos entering these equations are the ones obtained as solutions of the constraints.

In the end one must observe that the above considerations are true only for the case of manifolds with no boundary. Whenever boundary hypersurfaces occur, a much larger number of local invariants can be built and they contribute to the asymptotic expansion of the integrated heat kernel. This can be used to find appropriate generalizations of the work by Connes and other authors. At this moment it is still unclear what kind of boundary conditions (local or

non-local) form the most appropriate choice in simple or extended Euclidean supergravity. The boundary surfaces imply some subtleties in the case of supersymmetry and one expects that they would affect the constraints on the supermultiplets as well as those on the eigenspinors.

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